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Automatica 42 (2006) 1775 – 1781
www.elsevier.com/locate/automatica

Brief paper

Parameter bounds evaluation of Wiener models with noninvertible polynomial nonlinearities☆

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Received 9 June 2005; received in revised form 23 December 2005; accepted 10 May 2006
Available online 17 July 2006

Abstract

In this paper a three stage procedure is presented for deriving parameters bounds of SISO Wiener models when the nonlinear block is modeled by a possibly noninvertible polynomial and the output measurement errors are bounded. First, using steady-state input–output data, parameters of the nonlinear part are bounded by a tight orthotope. Then, given the estimated uncertain nonlinearity and the output measurements collected exciting the system with an input dynamic signal, bounds on the unmeasurable inner signal are computed. Finally, such bounds, together with noisy output measurements, are used for bounding the parameters of the linear block.

Keywords: Wiener systems; Identification; Bounded errors; Noninvertible polynomial nonlinearities

1. Introduction

Most physical systems are inherently nonlinear and, though in some cases they can be represented by linear models over a restricted operating range, only nonlinear representations are adequate for their description.

The nonlinear system considered in this paper, commonly referred to as Wiener model, is shown in Fig. 1; it consists of a linear dynamic system followed by static nonlinear block $\mathcal{N}$. The identification of such a model is carried out on the basis of the sequences u_t and y_t, while the inner signal x_t is not assumed to be available. In spite of its simplicity, such a model has been successfully used in many engineering fields, since it can embed process structure knowledge like, e.g., the presence of nonlinearity in the measurement equipment. Relevant applications of Wiener models can be found in a number of fields: adaptive signal processing (Wigren, 1998), echo cancellation (Treichler, Johnsson, & Larimore, 1987), blind adaptation (Godard, 1980; Wigren, 1997), harmonic signal modeling (Wigren & Handel, 1996), identification of biological systems (Hunter & Korenberg, 1986; Korenberg & Hunter, 1986), modeling of visual systems (den Brinker, 1989), modeling of distillation columns (Pearson & Pottmann, 2000; Zhu, 1999). The identification of Wiener models has attracted the attention of many authors (see, e.g., the survey paper Billings, 1980) exploiting a number of different techniques. Subspace identification is proposed in the contributions (Westwick & Verhaegen, 1996) and (Lovera, Gustafsson, & Verhaegen, 2000); maximum likelihood and recursive prediction error identification are, respectively, considered in (Hagenblad & Ljung, 1998) and (Wigren, 1993); frequency domain techniques are exploited in (Bai, 2003) and (Crama & Schoukens, 2001); a method based on nonparametric kernel regression estimation is proposed in (Greblicki, 1992) and a blind approach is taken in (Bai, 2002). The main difficulty in the identification of Wiener systems is that the internal signal is not available for measurement. However, under the assumption of invertible nonlinearities, which is a common one, the inner signal can be recovered from the output measurements through inversion of the previously estimated nonlinearity. Unfortunately, many output nonlinearities

☆ This paper was presented at 13th IFAC Symposium on System Identification, Rotterdam 2003. This paper was recommended for publication in revised form by Associate Editor Antonio Vicino under the direction of Editor Torsten Söderström.

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0005-1098/$ - see front matter
doi:10.1016/j.automatica.2006.05.010

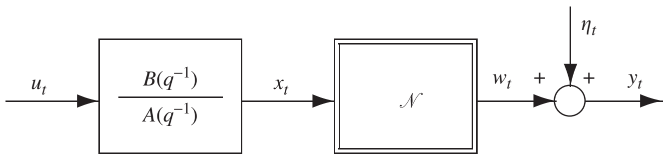

Fig. 1. Single-input single-output Wiener model.

encountered in real world problems are noninvertible (see, e.g., Wigren, 1998), thus the invertibility assumption appears to be quite restrictive. Removal of such an hypothesis makes the consistent evaluation of the inner signal sequence a difficult task even in the case of exactly known nonlinearities.

In all the papers mentioned above, the authors assume that the measurement error η_t is statistically described. A worthwhile alternative to the stochastic description of measurement errors are the bounded-errors characterization, where uncertainties are assumed to belong to a given set. In the bounding context, all parameter vectors belonging to the *feasible parameter set*, i.e. parameters consistent with the measurements, the error bounds and the assumed model structure, are feasible solutions of the identification problem. The interested reader can find further details on this approach in a number of survey papers (see, e.g., Milanese & Vicino, 1991). To our best knowledge, no contribution can be found which address the identification of Wiener models when the measurement error η_t is supposed to be bounded. In this paper we consider the identification of single-input single-output (SISO) Wiener models, when the nonlinear block can be modeled by a possible noninvertible polynomial, with finite and known degree, and when the output measurement errors are bounded.

2. Problem formulation

Consider the SISO discrete-time Wiener model shown in Fig. 1, where

$$x_t = \frac{B(q^{-1})}{A(q^{-1})}u_t. \tag{1}$$

$A(\cdot)$ and $B(\cdot)$ are polynomials in the backward shift operator q^{-1}, $(q^{-1}w_t = w_{t-1})$, $A(q^{-1}) = 1 + a_1q^{-1} + \cdots + a_{na}q^{-na}$ and $B(q^{-1}) = b_0 + b_1q^{-1} + \cdots + b_{nb}q^{-nb}$. The nonlinear block transforms x_t into the noise-free output w_t according to

$$w_t = \mathscr{N}(x_t, \gamma) = \sum_{k=1}^{n} \gamma_k x_t^k, \quad t = 1, \ldots, N, \tag{2}$$

where n is the polynomial degree and N is the length of the input sequence. In line with the work done by a number of authors, it is assumed that: (i) n is finite and a priori known; this hypothesis will be exploited in Propositions 1, 2, 5 and 6; (ii) the linear system is asymptotically stable (see, e.g., Krzyżak, 1993; Lang, 1993; Stoica & Söderström, 1982; Sun, Liu, & Sano, 1999); this is a standard hypothesis in open loop identification; (iii) $\sum_{j=0}^{nb} b_j \neq 0$, that is, the steady-state gain is not zero (see, e.g., Lang, 1993; Sun et al., 1999); (iv) an estimate of the process settling-time (see, e.g., Kalafatis, Wang, & Cluett, 1997) is available. Both hypotheses (iii) and (iv) will be exploited in the first stage of the proposed procedure when the estimation of the nonlinearity $\mathscr{N}$ is addressed.

Let y_t be the noise-corrupted measurements of w_t

$$y_t = w_t + \eta_t. \tag{3}$$

Measurements uncertainty is known to range within given bounds $\Delta\eta_t$, i.e.,

$$|\, \eta_t \,| \leqslant \Delta\eta_t. \tag{4}$$

Unknown parameter vectors $\gamma \in R^n$ and $\theta \in R^p$ are defined, respectively, as $\gamma^{\mathrm{T}} = [\gamma_1\ \gamma_2\ \ldots \gamma_n]$ and $\theta^{\mathrm{T}} = [a_1\ \ldots\ a_{na}\ b_0\ b_1\ \ldots b_{nb}]$, where $n_a + n_b + 1 = p$. It is well known that the parameterization of the structure of Fig. 1 is not unique. Here, it is assumed, without loss of generality, that the steady-state gain of the linear part be one. In this paper we address the problem of deriving bounds on parameters γ and θ consistently with given measurements, error bounds and the assumed model structure. The proposed solution is a three-stage procedure similar to the one proposed by the authors in (Cerone & Regruto, 2003) for the computation of parameter bounds for Hammerstein systems.

First stage: Exploiting M steady-state input–output data, one gets the feasible parameter set $\mathscr{D}_\gamma$ of the nonlinear block parameters, which is a convex polytope; then the central estimate $\gamma_j^{\mathrm{c}} = (\gamma_j^{\min} + \gamma_j^{\max})/2$ and the parameter uncertainty interval $[\gamma_j^{\min}, \gamma_j^{\max}]$ of each parameter γ_j are computed solving the following two linear programming problems:

$$\gamma_j^{\min} = \min_{\gamma \in \mathscr{D}_\gamma} \gamma_j, \quad \gamma_j^{\max} = \max_{\gamma \in \mathscr{D}_\gamma} \gamma_j. \tag{5}$$

Second stage: Given the estimated uncertain nonlinearity $\mathscr{N}(x_t, \gamma)$ and the output measurements collected exciting the system with an input dynamic signal, bounds on the inner signal x_t are computed.

Third stage: The bounds computed in the second stage, together with the input dynamic sequence, are used to obtain a polytopic outer approximation $\mathscr{D}'_\theta$ of the exact feasible parameter set $\mathscr{D}_\theta$ of the linear system. The central estimate $\theta_j^{\mathrm{c}} = (\theta_j^{\min} + \theta_j^{\max})/2$ and the parameter uncertainty interval $[\theta_j^{\min}, \theta_j^{\max}]$ of each parameter θ_j are computed solving the following two linear programming problems:

$$\theta_j^{\min} = \min_{\theta \in \mathscr{D}'_\theta} \theta_j, \quad \theta_j^{\max} = \max_{\theta \in \mathscr{D}'_\theta} \theta_j. \tag{6}$$

The first and the third stages of the procedure are quite standard and they will not be discussed in the paper. The interested readers can find the details in the previous works by the authors (Cerone & Regruto, 2003) and (Cerone, Milanese, & Regruto, 2003). The rest of the paper will focus on the novel contribution (the second stage of the procedure), i.e., the derivation of bounds on the inner unmeasurable signal through the partial inversion of the nonlinearity. The proof of all the Propositions presented in the paper can be found in (Cerone & Regruto, 2005).

The paper is organized as follows. In Section 3 we describe how to design a suitable input sequence to deal with the presence of a noninvertible polynomial nonlinearity at the output. The evaluation of the inner signal bounds is discussed in Section 4, while in Section 5 the computational aspects of quantities and sets involved in the estimation of the inner signal are analyzed. Finally, in Section 6 the proposed parameter bounding procedure is illustrated through a numerical example.

3. Dynamic experiment design

In the first stage of the parameter bounding procedure an uncertain description of the nonlinear block is obtained exploiting steady-state data. In order to estimate the parameters of the linear model in the third stage, one should first evaluate the inner signal $x_t \in R$ from the output records y_t of a dynamic experiment. Unfortunately, one must consider the fact that nonlinearity (2) is in general noninvertible, which means that, given the measured output y_t, the inner signal x_t cannot be evaluated uniquely. Nonuniqueness, unfortunately, is responsible for nonconsistent inner signal estimates. Given the feasible parameter set $\mathcal{D}_\gamma$ of the nonlinear block computed in the first stage of the procedure, the following families of polynomials can be defined:

$$\mathcal{V}_t = \{\mathcal{N}(x_t, \gamma) : \gamma \in \mathcal{D}_\gamma\} \tag{7}$$

and

$$\Pi_t = \{p_t(x_t, w_t, \gamma) : w_t \in R,\ \gamma \in \mathcal{D}_\gamma\}, \tag{8}$$

where

$$p_t(x_t, w_t, \gamma) = w_t - \sum_{k=1}^{n} \gamma_k x_t^k. \tag{9}$$

It is assumed that all polynomials in $\mathcal{V}_t$ and Π_t have degree equal to n, that is, $\gamma_n \neq 0\ \forall \gamma \in \mathcal{D}_\gamma$. In this case, in order to evaluate the inner signal x_t one has to find the real roots of the uncertain polynomial (8). Now, let us introduce the following definitions:

Definition 1. The set $W \subset R$ is an *output invertibility interval* for the uncertain polynomial $\mathcal{N}(x_t, \gamma)$ of degree n, if for $w_t \in W$ each polynomial $p_t(x_t, w_t, \gamma) \in \Pi_t$ shows either only one real root when n is odd or two real roots when n is even. Each w_t belonging to an Output Invertibility Interval is called an *invertible output value*.

Definition 2. The set $X \subset R$ is a *feasible inner-signal interval* for the Wiener system described by Eqs. (1) and (2) if the set of output values $\mathcal{O}=\{w_t \in R : w_t = \mathcal{N}(x_t, \gamma),\ \mathcal{N}(x_t, \gamma) \in \mathcal{V}_t,\ x_t \in X\}$ is an *output invertibility interval*.

The key idea exploited in this paper is to design an input sequence $\{u_t\}$ which forces the unmeasurable inner sequence $\{x_t\}$ to belong to a prescribed *feasible inner-signal interval X*. In such a way the corresponding output sequence $\{w_t\}$ belongs to an *output invertibility interval* of the polynomial $\mathcal{N}(x, \gamma)$ and each sample of the inner sequence $\{x_t\}$ can be bounded as described in Section 4. The rest of this Section will focus on how to design the input sequence $\{u_t\}$. The following two propositions provide a characterization of the output invertibility intervals and the feasible inner-signal intervals of the Wiener system described in Section 2.

Proposition 1. *The uncertain polynomial* $\mathcal{N}(x_t, \gamma)$ *with* $\gamma \in \mathcal{D}_\gamma$, *shows the following two output invertibility intervals*:

$$\overline{W} =]\overline{w}, +\infty[\quad \text{and} \quad \underline{W} =]-\infty, \underline{w}[\ \text{for } n \ \text{ odd} \tag{10}$$

$$\overline{W} =]\overline{w}, +\infty[\ \text{for } n \text{ even}, \quad \gamma_n > 0 \tag{11}$$

and

$$\underline{W} =]-\infty, \underline{w}[\ \text{for } n \text{ even}, \quad \gamma_n < 0, \tag{12}$$

where

$$\overline{w} = \max_{x_t \in \Upsilon_t} \max_{\gamma \in \mathcal{D}_\gamma} \sum_{k=1}^{n} \gamma_k x_t^k, \quad \underline{w} = \min_{x_t \in \Upsilon_t} \min_{\gamma \in \mathcal{D}_\gamma} \sum_{k=1}^{n} \gamma_k x_t^k, \tag{13}$$

$$\Upsilon_t = \left\{ x_t \in R : \frac{\mathrm{d}}{\mathrm{d}x_t} \sum_{k=1}^{n} \gamma_k x_t^k = 0,\ \text{for some } \gamma \in \mathcal{D}_\gamma \right\}. \tag{14}$$

Proposition 2. *The Wiener system described by Eqs.* (1) *and* (2), *with uncertain output polynomial* $\mathcal{N}(x_t, \gamma)$, *shows the following Feasible inner-signal intervals*:

$$\overline{X} =]\overline{x}, +\infty[, \quad \underline{X} =]-\infty,\ \underline{x}[, \tag{15}$$

where

$$\overline{x} = \max \left\{ x_t \in R : \frac{1 + \mathrm{sign}(\gamma_n)}{2}\overline{w} + \frac{1 - \mathrm{sign}(\gamma_n)}{2}\underline{w} - \sum_{k=1}^{n} \gamma_k x_t^k = 0,\ \text{for some } \gamma \in \mathcal{D}_\gamma \right\}, \tag{16}$$

$$\underline{x} = \min \left\{ x_t \in R : \frac{1 + (-1)^n \mathrm{sign}(\gamma_n)}{2}\overline{w} - \sum_{k=1}^{n} \gamma_k x_t^k + \frac{1 - (-1)^n \mathrm{sign}(\gamma_n)}{2}\underline{w} = 0,\ \text{for some } \gamma \in \mathcal{D}_\gamma \right\}. \tag{17}$$

A graphical illustration of Propositions 1 and 2 is depicted in Fig. 2 for the case of an odd polynomial of degree 3.

3.1. Input sequence design

In order to drive the inner signal $\{x_t\}$ into the desired interval X, the input signal $\{u_t\}$ should contain a DC component u_{DC} (offset) and a dynamic exciting signal $\{u_{\mathrm{td}}\}$ whose amplitudes should be chosen in such a way that $x_t = x_{\mathrm{DC}} + x_{\mathrm{td}}$ belongs to $X\ \forall t$. Since the steady-state gain of the linear subsystem is constrained to be one, the amplitudes of the DC components in $u_t = u_{\mathrm{DC}} + u_{\mathrm{td}}$ and x_t are the same, i.e., $u_{\mathrm{DC}} = x_{\mathrm{DC}}$. Guidelines

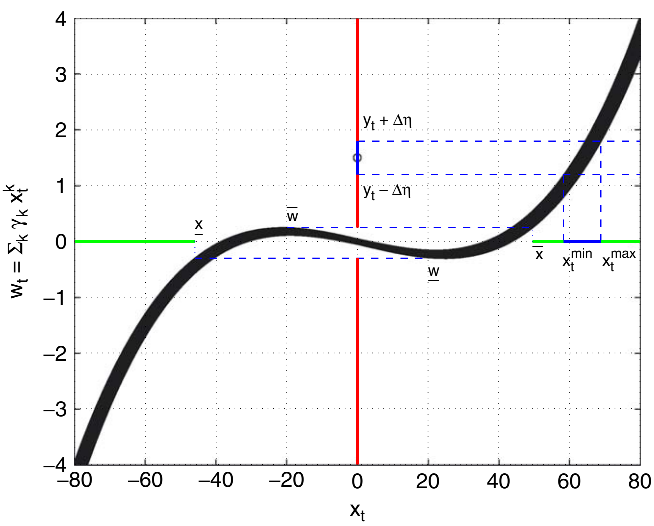

Fig. 2. Output invertibility intervals (red), feasible inner-signal intervals (green) and inner signal bounds of an uncertain odd polynomial.

for the design of the dynamic exciting signal $\{u_{\mathrm{td}}\}$ are provided by the following two propositions.

Proposition 3. *For a given* $u_{\mathrm{DC}} \geqslant \overline{x}$, *each sample of the sequence* $\{x_t\}$ *belongs to* $\overline{X}$ *if*

$$\|\{u_{\mathrm{td}}\}\|_\infty \leqslant \frac{|u_{\mathrm{DC}} - \overline{x}|}{h_{\mathrm{up}}}. \tag{18}$$

where h is the impulse response of the linear block and h_{up} *is an upper bound of its* ℓ_1 *norm*; $\|\cdot\|_\infty$ *is the* ℓ_∞ *norm of a sequence.*

Proposition 4. *For given* $u_{\mathrm{DC}} \leqslant \underline{x}$, *each sample of the sequence* $\{x_t\}$ *belongs to* $\underline{X}$ *if*

$$\|\{u_{\mathrm{td}}\}\|_\infty \leqslant \frac{|u_{\mathrm{DC}} - \underline{x}|}{h_{\mathrm{up}}}. \tag{19}$$

When no a priori information on the ℓ_1-norm of the linear system is available, the following results can be exploited.

Proposition 5. *All the samples of the output sequence* $\{w_t\}$ *belong to the same output invertibility interval* W (*either* $W = \overline{W}$ *or* $W = \underline{W}$) *if the samples of the corresponding measured sequence* $\{y_t\}$ *satisfy the following inequalities, where* $\overline{y}_t = \overline{w} + \Delta\eta_t$ *and* $\underline{y}_t = \underline{w} - \Delta\eta_t$:

$$y_t > \overline{y}_t \ \forall t \quad \text{or } y_t < \underline{y}_t \ \forall t, \quad \text{when } n \text{ is odd} \tag{20}$$

$$\operatorname{sign}(\gamma_n)(y_t - \operatorname{sign}(\gamma_n)\Delta\eta_t) > \frac{1+\operatorname{sign}(\gamma_n)}{2}\overline{y}_t - \frac{1-\operatorname{sign}(\gamma_n)}{2}\underline{y}_t, \quad \forall t, \quad \text{when } n \text{ is even}. \tag{21}$$

Proposition 5 provides sufficient conditions for $\{w_t\}$ to belong either to $\overline{W}$ or to $\underline{W}$. Thus, when no a priori information on the ℓ_1-norm of the linear systems is available, the condition $x_t \in X \forall t$ can be indirectly satisfied varying the amplitude of the dynamic sequence $\{u_{\mathrm{td}}\}$ by trial and error until the measured output sequence $\{y_t\}$ satisfies either condition (20) or (21).

4. Evaluation of bounds on the inner signal

Given the estimated uncertain polynomial nonlinearity $\mathscr{V}_t$ and a sequence of measured outputs $\{y_t\}$, obtained exciting the Wiener system with the input sequence $\{u_t\}$ designed as described in Section 3, in this section it is shown how upper and lower bounds on the samples of the unmeasurable inner signal x_t can be evaluated.

The following proposition provides bounds for the case $\gamma_n > 0$ and $X = \overline{X}$. Similar propositions for the other cases are not reported since they are only slight variations of this result.

Proposition 6. *Given the estimated polynomial nonlinearity* $\mathscr{N}(x_t, \gamma)$ *with* $\gamma \in \mathscr{D}_\gamma$ *and* $\gamma_n > 0$, *an input sequence* $\{u_t\}$ *which drives the inner unmeasurable signal into a feasible inner-signal interval* $\overline{X}$, *and the corresponding measured output sequence* $\{y_t\}$, *each sample* x_t *of the inner sequence* $\{x_t\}$ *is bounded as follows*:

$$x_t^{\min} \leqslant x_t \leqslant x_t^{\max}, \tag{22}$$

$$x_t^{\max} = \max\left\{x_t \in \overline{X} : y_t + \Delta\eta_t - \sum_{k=1}^{n} \gamma_k x_t^k = 0, \ \textit{for some } \gamma \in \mathscr{D}_\gamma\right\}, \tag{23}$$

$$x_t^{\min} = \max\{\overline{x}, \hat{x}_t^{\min}\},$$

$$\hat{x}_t^{\min} = \min\left\{x_t \in R : y_t - \Delta\eta_t - \sum_{k=1}^{n} \gamma_k x_t^k = 0, \ \textit{for some } \gamma \in \mathscr{D}_\gamma\right\}. \tag{24}$$

A graphical illustration of Proposition 6 is shown in Fig. 2 for an uncertain odd polynomial of degree 3.

5. Computational algorithms

In this section the computational aspects of quantities and sets involved in the estimation of the inner signal are analyzed.

5.1. Computation of Υ_t

First consider the set defined by Eq. (14), i.e., the set of real valued x_t for which the uncertain polynomial shows stationary points. The first derivative of the uncertain polynomial is still an uncertain polynomial, namely

$$p_t{}'(x_t,\gamma) = -\frac{\mathrm{d}}{\mathrm{d}x_t}\sum_{k=1}^{n}\gamma_k x_t^k = -\sum_{k=1}^{n} k\gamma_k x_t^{k-1} \tag{25}$$

which, clearly, shows nonlinear relations in the unknown x_t and the uncertain γ. It is noticed that given an $x_t \in R$, it belongs to the real spectral set of polynomial (25) if and only if there exists at least one $\gamma \in \mathscr{D}_\gamma$ such that x_t is the solution of the equation $\sum_{k=1}^{n} k\gamma_k x_t^{k-1} = 0$. In order to find the real roots of (25), a one-dimensional gridding on the variable x_t is proposed. For each grid point x_t one must check if there exists a solution to a set of $2M$ linear inequalities (i.e., $\gamma \in \mathscr{D}_\gamma$) and one linear equality (i.e., $\sum_{k=1}^{n} k\gamma_k x_t^{k-1}=0$) in the unknown $\gamma \in R^n$. If a solution γ exists, then x_t is a real roots of the uncertain polynomial (25). Such a test can be performed solving a linear programming problem (see, e.g., Schrijver, 1986).

5.2. Computation of $\overline{w}$ and $\underline{w}$

Next Eq. (13) which defines two nonlinear programming problems is considered. We note that when x_t is given, problems (13) simplify to linear programs. Thus, to compute $\overline{w}$ and $\underline{w}$, for each value of $x_t \in \Upsilon_t$, the solution of two linear programming problems with n variables and $2M$ constraints is required. A one-dimensional gridding procedure is used in order to carry out the optimization over a finite number of $x_t \in \Upsilon_t$.

5.3. Computation of $\overline{x}$ and $\underline{x}$

Here Eq. (16) and Eq. (17) are considered. In order to simplify the discussion, odd degree polynomial with $\gamma_n > 0$ only are considered since similar considerations can be made in all other cases ($\gamma_n > 0$, $\gamma_n < 0$, n odd, n even). In this case one gets

$$\overline{x} = \max\{x_t : p_t(x_t,\gamma,\overline{w}) = 0,\ \text{for some } \gamma \in \mathscr{D}_\gamma\}, \tag{26}$$

$$\underline{x} = \min\{x_t : p_t(x_t,\gamma,\underline{w}) = 0,\ \text{for some } \gamma \in \mathscr{D}_\gamma\}. \tag{27}$$

As a matter of fact Eqs. (26) and (27) show nonlinear relations in the unknown x_t and the uncertain γ. The following notes can be made in order to develop an algorithm for the computation of $\overline{x}$:

(a) if x_t is the solution of problem (26), then the set $\Gamma_t(x_t,\gamma,\overline{w}) = \{\gamma \in \mathscr{D}_\gamma : p_t(x_t,\gamma,\overline{w}) = 0\}$ is not empty;

(b) let us consider the nominal $p_t^{\text{nom}}(x_t,\gamma^*,\overline{w})$ obtained, e.g., setting $\gamma^* = \gamma^c$; it is noticed that only right side of the maximum real root of equation $p_t^{\text{nom}}(x_t,\gamma^*,\overline{w}) = 0$ has to be explored in order to find a suitable approximation of $\overline{x}$.

Stringing together notes (a) and (b) the following algorithm is proposed for the approximate computation of $\overline{x}$.

Algorithm 1 (Computation *of* $\overline{x}$).

1. Set $\alpha = \alpha_0$ **and** $\varepsilon \triangleq$ **prescribed tolerance**.
2. Compute $r = \max\{x_t \in R : p_t^{\text{nom}}(x_t,\gamma^c,\overline{w}) = 0\}$.
3. Set $x_m = r$.
4. Set $x_M = x_m + \alpha$.
5. If $\exists \gamma^\diamond \in \mathscr{D}_\gamma : p_t(x_M,\gamma^\diamond,\overline{w}) = 0$ **then**
    $x_m = x_M$;
  else
    If $|x_M - x_m| < \varepsilon$ **then**
      $\overline{x}_* = x_M$;
      return $\overline{x}_*$;
      stop algorithm.
    else
      $\alpha = \alpha/2$;
    end if
  end if.
8. Repeat from 4.

The main properties of Algorithm 1 are highlighted by the following proposition.

Proposition 7. *Algorithm 1 enjoys the following properties*:
(1) *Algorithm 1 is convergent.*
(2) *Algorithm 1 provides an upper bound* $\overline{x}_*$ *of* $\overline{x}$; *the absolute errors of such a bound is bounded by* ε.
(3) *The check required by step 5 of Algorithm 1 can be performed solving a linear programming problem* (see, e.g., Schrijver, 1986).

Similar results can be obtained for the computation of $\underline{x}$ which can be computed with a slight modification of Algorithm 1 (see Cerone & Regruto, 2005).

5.4. Computation of $x_t^{\max}$ and $x_t^{\min}$

Finally, the computation of the inner signal bounds is considered. In this case one must compute

$$x_t^{\max} = \max\{x_t \in \overline{X} : p_t(x_t, y_t + \Delta\eta_t, \gamma) = 0,\ \text{for some } \gamma \in \mathscr{D}_\gamma\} \tag{28}$$

$$x_t^{\min} = \max\{\overline{x}, \hat{x}_t^{\min}\}, \tag{29}$$

where $\hat{x}_t^{\min} = \min\{x_t \in R : p_t(x_t, y_t - \Delta\eta_t, \gamma) = 0,\ \text{for some } \gamma \in \mathscr{D}_\gamma\}$. From (28) can be seen that $x_t^{\max}$ can be computed using Algorithm 1 simply substituting $p_t(x_M,\overline{w},\gamma)$ with $p_t(x_M, y_t + \Delta\eta_t, \gamma)$. $\hat{x}_t^{\min}$ can be computed using the slight modification of Algorithm 1 used to compute $\underline{x}$ simply substituting $p_t(x_M,\underline{w},\gamma)$ with $p_t(x_M, y_t - \Delta\eta_t, \gamma)$ (Cerone & Regruto, 2005).

6. A simulated example

The system considered here is characterized by $\gamma = [\gamma_1\ \gamma_2\ \gamma_3]^{\mathrm{T}} = [-5\ \ -4\ \ 1]^{\mathrm{T}}$ and $\theta = [a_1\ a_2\ b_1\ b_2]^{\mathrm{T}} = [-1.1\ 0.28\ 0.1\ 0.08]^{\mathrm{T}}$. The considered nonlinear function is an odd noninvertible polynomial. From the simulated transient sequence $\{w_t, \eta_t\}$ and steady-state data

Table 1
Nonlinear block parameter central estimates (γ_j^c) and parameter uncertainty bounds ($\Delta\gamma_j$) against signal to noise ratio ($\overline{SNR}$)

$\overline{SNR}$ (dB)	γ_j	True value	γ_j^c	$\Delta\gamma_j$
58.2	γ_1	−5.000	−4.999	2.1e−3
	γ_2	−4.000	−4.000	1.8e−4
	γ_3	1.000	1.000	4.8e−5
38.2	γ_1	−5.000	−5.027	3.6e−2
	γ_2	−4.000	−3.995	8.1e−3
	γ_3	1.000	1.001	1.6e−3
28.6	γ_1	−5.000	−5.040	8.2e−2
	γ_2	−4.000	−4.003	6.2e−3
	γ_3	1.000	1.000	1.9e−3
18.4	γ_1	−5.000	−5.101	1.1e−1
	γ_2	−4.000	−4.000	1.0e−2
	γ_3	1.000	1.004	5.1e−3

Table 2
Linear system parameter central estimates (θ_j^c) and parameter uncertainty bounds ($\Delta\theta_j$) against signal to noise ratio (SNR) when $N = 100$

$\overline{SNR}$ (dB)	θ_j	True value	θ_j^c	$\Delta\theta_j$
58.2	θ_1	−1.100	−1.100	5.3e−3
	θ_2	0.280	0.280	5.1e−3
	θ_3	0.100	0.100	6.1e−4
	θ_4	0.080	0.080	5.6e−4
38.0	θ_1	−1.100	−1.106	7.9e−2
	θ_2	0.280	0.288	7.4e−2
	θ_3	0.100	0.100	8.0e−3
	θ_4	0.080	0.081	9.0e−3
28.3	θ_1	−1.100	−1.155	2.1e−1
	θ_2	0.280	0.331	2.0e−1
	θ_3	0.100	0.105	2.0e−2
	θ_4	0.080	0.074	2.9e−2
18.2	θ_1	−1.100	−1.211	3.9e−1
	θ_2	0.280	0.403	3.6e−1
	θ_3	0.100	0.099	4.2e−2
	θ_4	0.080	0.101	4.7e−2

Table 3
Linear system parameter central estimates (θ_j^c) and parameter uncertainty bounds ($\Delta\theta_j$) against signal to noise ratio (SNR) when $N = 1000$

SNR (dB)	θ_j	True value	θ_j^c	$\Delta\theta_j$
58.2	θ_1	−1.100	−1.100	1.9e−3
	θ_2	0.280	0.280	1.8e−3
	θ_3	0.100	0.100	1.9e−4
	θ_4	0.080	0.080	2.2e−4
38.4	θ_1	−1.100	−1.102	5.8e−2
	θ_2	0.280	0.282	5.4e−2
	θ_3	0.100	0.100	6.1e−3
	θ_4	0.080	0.079	5.9e−3
28.2	θ_1	−1.100	−1.106	8.9e−2
	θ_2	0.280	0.284	8.2e−2
	θ_3	0.100	0.099	8.7e−3
	θ_4	0.080	0.080	1.0e−2
18.2	θ_1	−1.100	−1.113	1.5e−1
	θ_2	0.280	0.293	1.4e−1
	θ_3	0.100	0.101	1.6e−2
	θ_4	0.080	0.078	1.7e−2

$\{\bar{w}_s, \bar{\eta}_s\}$, the signal to noise ratios (SNR) are evaluated, respectively, through $SNR = 10\log\{\sum_{t=1}^{N} w_t^2/\sum_{t=1}^{N} \eta_t^2\}$ and $\overline{SNR} = 10\log\{\sum_{s=1}^{M} \bar{w}_s^2/\sum_{s=1}^{M} \bar{\eta}_s^2\}$.

Bounded absolute output errors have been considered when simulating the collection of both steady state data, $\{\bar{u}_s, \bar{y}_s\}$, and transient sequence $\{u_t, y_t\}$. Here we assumed $|\ \eta_t\ | \leqslant \Delta\eta_t$ and $|\ \bar{\eta}_s\ | \leqslant \Delta\bar{\eta}_s$ where η_t and $\bar{\eta}_s$, are random sequences belonging to the uniform distributions $U[-\Delta\eta_t, +\Delta\eta_t]$ and $U[-\Delta\bar{\eta}_s, +\Delta\bar{\eta}_s]$, respectively. Bounds on steady-state and transient output measurement errors were supposed to have the same value, i.e., $\Delta\eta_t = \Delta\bar{\eta}_s \triangleq \Delta\eta$, and were chosen in such a way as to simulate four different values of SNR at the output, namely 60, 40, 30 and 20 dB. For a given $\Delta\eta$, the length of steady-state and the transient data are $M = 10$ and $N = [100, 1000]$, respectively. The steady-state input sequence $\{\bar{u}_s\}$ belongs to the interval $[-2, +2]$, while the transient input sequence $\{u_t\}$ belongs to the uniform distribution $U[-2, +2]$. Results about the nonlinear and the linear block are reported in Table 1 and Tables 2 and 3, respectively. For low noise level ($\overline{SNR}$=60 dB) and for all N, the central estimates of both the nonlinear static block and the linear model are consistent with the true parameters. For higher noise level ($\overline{SNR} \leqslant 40$ dB), both γ^c and θ^c give satisfactory estimates of the true parameters. As the number of observations increases (from $N = 100$ to 1000), parameter uncertainty bounds $\Delta\gamma_j$ and $\Delta\theta_j$ decreases, as expected.

7. Concluding remarks

In this paper the identification of SISO Wiener models has been considered when the nonlinear block can be modeled by a polynomial, with finite and known degree, and when the output measurements are corrupted by unknown but bounded noise. The proposed solution is a three stage procedure similar to the one proposed by the authors in a previous work for the computation of parameter bounds for Hammerstein systems. Firstly, using steady-state input–output data, parameters of the nonlinear block are tightly bounded. Secondly, given the estimated uncertain nonlinearity and the output measurements collected exciting the system with an input dynamic signal, bounds on the unmeasurable inner signal are computed. Thirdly, such bounds, together with the input dynamic sequence, are used to obtain a polytopic outer approximation of the exact feasible parameter set of the linear system. The main contribution of the paper is the second stage of the procedure, i.e., the derivation of bounds on the inner unmeasurable signal through the partial inversion of the polynomial nonlinearity. Current limitations of the approach, and possible directions for further research, are as follows.

- The proposed procedure is based on the partial inversion of the nonlinearity performed through the characterization of a suitable invertibility region for the polynomial. Thus, the applicability of such a procedure is limited to Wiener systems with noninvertible polynomial nonlinearities which have at least one Output Invertibility Interval in their normal operating range.
- The trial and error method proposed in Proposition 5, which has to be used when not even an rough upper bound on the ℓ_1 norm of the linear block is known, could be, in general, time consuming.
- Even though the proposed approach is computationally tractable, the complexity is high. That is mainly due to the use of linear gridding which can also affect the accuracy of the results. Thus, some care must be taken in the algorithms implementation.

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